Now under the assumption of semi-incompressibility of the fluid, one has

$$\rho - \rho_\infty = -\rho\beta\theta \tag{6}$$

where β is the coefficient of thermal expansion, and ρ_∞ and θ denote density and temperature difference of the fluid far away from the plate, respectively; otherwise ρ is treated to be constant. On considering this fact, Eq. (5) gives

$$v = \text{const} = v_0 \tag{7}$$

and thereby reduces Eqs. (3) and (4) to

$$\rho v_0(du/dy) = -(\partial p/\partial x) + (d\sigma_y{}^x/dy) - \rho g \tag{8}$$

$$0 = -(\partial p/\partial y) + (d\sigma_y{}^y/dy) \tag{9}$$

On using (7), Eq. (1) gives

$$\sigma_x{}^x + \tau[v_0(d\sigma_x{}^x/dy) - 2\sigma_y{}^y(du/dy)] = 0 \tag{10}$$

$$\sigma_y{}^x + \tau[v_0(d\sigma_y{}^x/dy) - \sigma_y{}^y(du/dy)] = \mu(du/dy) \tag{11}$$

$$\sigma_y{}^y + \tau[v_0(d\sigma_y{}^y/dy)] = 0 \tag{12}$$

Equation (12) shows that $\sigma_y{}^y = 0$ is a particular solution of this equation. Hence, putting $\sigma_y{}^y = 0$ in Eq. (11), one gets

$$\sigma_y{}^x + \tau v_0(d\sigma_y{}^x/dy) = \mu(du/dy) \tag{13}$$

From Eq. (8), $\partial p/\partial x$ is constant, since remaining terms in the equation are independent of x, and $\partial^2 p/\partial y\partial x = 0$, by virtue of Eq. (9). Therefore,

$$\partial p/\partial x = \text{const} = -\rho_\infty g \tag{14}$$

Putting this value of $\partial p/\partial x$ in Eq. (8) and using Eq. (6), one has

$$v_0(du/dy) = \beta\theta g + (1/\rho)(d\sigma_y{}^x/dy) \tag{15}$$

Eliminating $\sigma_y{}^x$ between Eqs. (13) and (15), one gets

$$[(\mu/\rho) - \tau v_0{}^2](d^2u/dy^2) - v_0(du/dy) + \beta g\theta + \tau\beta g v_0(d\theta/dy) = 0 \tag{16}$$

The equation of energy in the mechanical units can be written as

$$\rho c v_0(d\theta/dy) = k(d^2\theta/dy^2) + \mu(du/dy)^2 \tag{17}$$

where c and k denote the specific heat and thermal conductivity of the fluid, respectively.

The rigorous solution of the key Eqs. (16) and (17), in general, can be obtained numerically. However, the analysis then would be very complicated, and the physical aspect of the problem would be masked. Therefore, neglecting the viscous dissipation term in Eq. (17), which is justified for slow motion as in the case with free-convection flows, one gets

$$\rho c\, v_0(d\theta/dy) = k(d^2\theta/dy^2) \tag{18}$$

Equations (16) and (18) now are solved under the boundary conditions

$$\begin{array}{ll} u = 0 \text{ at } y = 0 & u = U_\infty \text{ at } y = \infty \\ \theta = \theta_0 \text{ at } y = 0 & \theta = 0 \text{ at } y = \infty \end{array} \tag{19}$$

Before solving these equations, it is interesting to remark that the solution of Eqs. (16) and (18) subject to boundary conditions (19) is physically possible only if $v_0 < 0$ (suction) and $\mu/\rho > \tau v_0{}^2$. That the solution also appears in case of fluid injection ($v_0 > 0$) provided $\mu/\rho < \tau v_0{}^2$, as pointed out by Gupta in his case, does not hold in this case. This is so because the solution of Eq. (18) for $v_0 > 0$ gives positive exponentials.

Taking $v_0 < 0$, it is found that the solution of Eq. (18) subject to (19) is

$$\theta = \theta_0\, e^{-(\rho c v_0/k)y} \tag{20}$$

Eliminating θ between (16) and (20) (taking $v_0 < 0$), one gets

$$(1 - \alpha)(d^2u/d\sigma^2) + (du/d\eta) + A[1 + \sigma\alpha]e^{-\sigma\eta} = 0 \tag{21}$$

where $\eta = \rho v_0 y/\mu$, $\alpha = \tau v_0{}^2\rho/\mu$, $A = \mu g\theta_0\beta/\rho v_0{}^2$, and $\sigma = \mu c/k$ is the Prandtl number.

The solution of Eq. (21) subject to (19) is found to be

$$u = U_\infty[1 - e^{-[\eta/(1-\alpha)]}] - \frac{A(1 + \sigma\alpha)}{[(1 - \alpha)\sigma^2 - \sigma]}\,[e^{-\sigma\eta} - e^{-[\eta/(1-\alpha)]}] \tag{22}$$

Equation (22) reduces to the familiar asymptotic suction profile of Meredith and Griffith[4] as α and A tend to zero in the limit. It also is seen from the solution that increasing A decreases u, whereas increasing σ decreases the effect of free convection. That this will be so is intuitively clear but only partially, since various factors affect the flow pattern, and hence mathematical corroboration has some interest. Another point that seems to be of physical interest is that the effect of elasticity of the fluid τ will not be perceptible unless v_0 (suction) is present. The reason is that in steady motion the fluid elements do not undergo any change in their state of stress. But as soon as suction comes in, the fluid elements move from one layer of the fluid to the other and thus experience a change in their stress state. The preceding remark, however, will not hold true if in Eq. (11) the third term on the left-hand side, which is an extra term over the material derivative of $\sigma_y{}^x$, also is taken into account. Since in this analysis the author has taken $\sigma_y{}^y = 0$, this term becomes ineffective and the remaining terms correspond to material derivatives only.

The shear stress Tw at the wall is given in the form

$$\frac{Tw}{\rho U_\infty v_0} = \frac{1}{1 - \alpha} + \frac{\sigma A(1 + \sigma\alpha)}{U_\infty[(1 - \alpha)\sigma^2 - \sigma]} \tag{23}$$

which clearly indicates that skin friction increases with the increase of A, as was to be expected.

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Convergence Technique for the Steepest-Descent Method of Trajectory Optimization

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THE steepest-descent approach to trajectory optimization, developed by Kelley[1] and Bryson et al.,[2, 3] has proved to be quite successful in overcoming the two-point boundary value problems associated with the calculus of variations. The steepest-descent method is an iterative procedure, requiring repeated forward and backward solutions of sets of differential equations. It is thus of interest to consider techniques for speeding convergence of the iterative process to the optimum trajectory. In this note, a technique re-

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sembling a closed-loop guidance system is presented. This method has provided rapid convergence for a variety of missions.

The optimization problem can be described as follows. Given a system described by the set of differential equations

$$\dot{x}_i = f_i[x_1, \ldots, x_n, \alpha(t), t] \qquad i = 1, \ldots n \tag{1}$$

with initial conditions specified at t_0, find the control variable history, $\alpha(t)$, which maximizes a pay-off function, $\phi(x_1, \ldots, x_n)$, subject to a constraint $\psi(x_1, \ldots, x_n) = 0$, where ϕ and ψ are both evaluated at a terminal time T. More than one control variable and terminal constraint could be specified, but that would only complicate the algebra.

The steepest-descent procedure is started by guessing a control history $\alpha(t)$ and by solving the set of Eqs. (1) to determine a nominal trajectory. This trajectory, in general, will not be optimum and will not satisfy the terminal constraint. The effects of small changes in the control and trajectory variables on the terminal quantities ϕ and ψ are then determined. This is done by solving a set of differential equations which are adjoint to the linear perturbation equations written about the nominal trajectory. Initial conditions for the adjoint equations are specified at the terminal time T. Integration of the equations therefore proceeds backward in time. Expressions of the form

$$\delta\phi(T) = \int_{t_0}^{T} \Lambda_\phi \delta\alpha dt + \left(\sum_{i=1}^{n} \lambda_{\phi i}\delta x_i\right)_{t=t_0} \tag{2}$$

and

$$\delta\psi(T) = \int_{t_0}^{T} \Lambda_\psi \delta\alpha dt + \left(\sum_{i=1}^{n} \lambda_{\psi i}\delta x_i\right)_{t=t_0} \tag{3}$$

are obtained. $\delta\alpha$ and δx_i are the perturbations in the control and trajectory variables, and the λ's are the solutions of the adjoint equations. Λ is given by the equation

$$\Lambda = \sum_{i=1}^{n} \lambda_i \frac{\partial f_i}{\delta\alpha} \tag{4}$$

One now seeks the function $\delta\alpha(t)$, which will produce a $\delta\psi(T)$ equal to the negative of the error in ψ on the nominal trajectory while simultaneously giving a specified improvement $\delta\phi(T)$ in the pay-off quantity. Furthermore, the amount of $\delta\alpha$ required to accomplish this should be as small as possible, in an integral square sense. The function that satisfies these requirements is

$$\delta\alpha(t) = K_\phi \Lambda_\phi + K_\psi \Lambda_\psi \tag{5}$$

where the K's are determined by substituting Eq. (5) into Eqs. (2) and (3) and have the values

$$K_\phi = \frac{I_{\psi\psi}\Delta\phi - I_{\phi\psi}\Delta\psi}{I_{\phi\phi}I_{\psi\psi} - I_{\phi\psi}^2} \tag{6}$$

$$K_\psi = \frac{I_{\phi\phi}\Delta\psi - I_{\phi\psi}\Delta\phi}{I_{\phi\phi}I_{\psi\psi} - I_{\phi\psi}^2} \tag{7}$$

where

$$\Delta\phi = \delta\phi(T) - \left(\sum_{i=1}^{n} \lambda_{\phi i}\delta x_i\right)_{t=t_0}$$

and

$$I_{\phi\phi} = \int_{t_0}^{T} \Lambda_\phi^2 dt \qquad I_{\phi\psi} = \int_{t_0}^{T} \Lambda_\phi \Lambda_\psi dt$$

Note that in the references cited, the K's are constants which are computed once only at time t_0. With fixed initial conditions, the summation terms are therefore zero. The time history of $\delta\alpha$ obtained from Eq. (5) is added to the nominal α, and Eqs. (1) are solved again to determine a new nominal trajectory. The new nominal is then used for the backward solution of the adjoint equations. This process is repeated until no further improvement in $\delta\phi$ can be obtained.

The speed of convergence to the optimum solution is strongly dependent upon how close the original nominal is to the optimum. This is true because the change from trajectory to trajectory cannot be too large without violating the assumptions of linear perturbation theory made in deriving the expressions for $\delta\phi$ and $\delta\psi$. If too large a perturbation in α is allowed, the values of the terminal conditions obtained after integrating Eqs. (1) will differ radically from those predicted by Eqs. (2) and (3).

The distinctive feature of the convergence scheme presented here is that each forward integration of the system equations is regarded as a closed-loop guidance problem. The K's, which may be considered as variable guidance parameters, are recomputed continuously, taking into account the deviation from the nominal trajectory at all times. The time t_0 becomes a running variable which is equal to the current time, and the summation terms involved in Eqs. (6) and (7) are no longer zero. The virtue of this closed-loop approach is that larger deviations from the nominal trajectory can be tolerated while still meeting the desired terminal conditions. It is therefore possible to move more rapidly toward the optimum trajectory. It should be noted that this method of guiding to desired terminal conditions is identical to the lambda-matrix control scheme described in Ref. 4.

In using this convergence method, no attempt is made to improve the pay-off quantity on the first forward integration. The control variable is selected to satisfy the terminal constraints with the pay-off quantity left free. This is done by setting

$$\delta\alpha = K_\psi \Lambda_\psi \tag{8}$$

and substituting in Eq. (3) to determine K_ψ. The reason for concentrating on meeting terminal constraints is that changes in these constraints will often have a greater influence on the pay-off quantity than will changes in the control variable. Thus, when one asks for a large change in the constraints in order to meet desired terminal values, it is very difficult to pick a value for the pay-off quantity that can be obtained with a sensible control program. The safest thing to do is simply to leave the pay-off unconstrained. It is interesting to observe that, in the case of vehicles for which thrust is large compared to gravity and drag forces, any reasonable control program that satisfies terminal constraints will provide close to the optimum pay-off. The control program itself, however, may be far from the exact optimum.

The size of the improvement in pay-off to be asked for initially is specified in the computer input. After integrating the system equations with the new control program, the terminal values of the pay-off and the constraints are compared with the desired values. If any improvement in pay-off is obtained, relative to the last nominal, without seriously violating the terminal constraints, this new trajectory becomes the nominal, and the same improvement is asked for once again. If no pay-off improvement is obtained, or if any improvement is accompanied by a serious violation of the terminal constraints, the trajectory is not used. Instead, the improvement asked for is cut in half, and a new trajectory is computed. The process terminates when the desired improvement becomes less than a preset number.

The following simple example illustrates the behavior of this convergence scheme as observed in a number of different missions. A thrusting vehicle moves on a horizontal, frictionless plane. The initial direction, velocity, and mass are specified. Determine the thrust attitude history that will maximize the terminal mass while slowing down to a given velocity. The lateral deviation at the terminal velocity, measured with respect to initial direction, is to be zero. The pay-off quantity, therefore, is mass, and the constraint is on lateral deviation. The answer to this problem, of

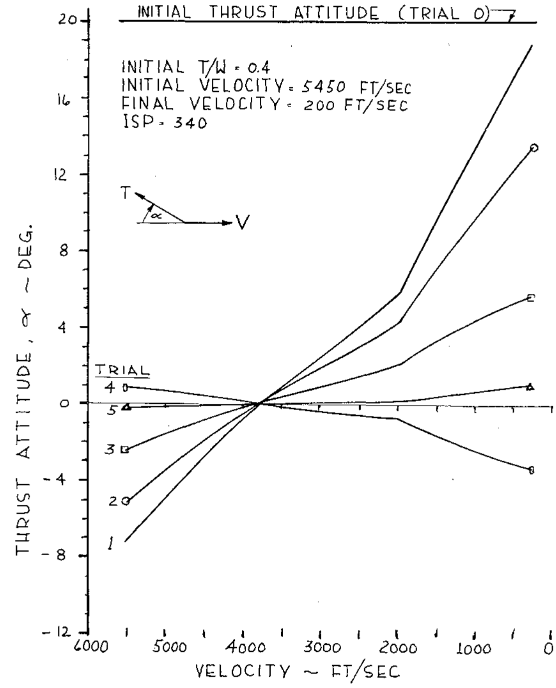

Fig. 1 Thrust attitude on successive optimization trials

Table 1 Pay-off and lateral deviation on successive trials

Trial	Mass ratio / Optimum mass ratio	Lateral deviation
0	1.03125	165,000
1	1.00497	−806
2	1.00255	67
3	1.00050	90
4	1.00014	41
5	1.00001	−6

course, is that the vehicle should move on a straight line with the thrust pointed in a direction opposite to the velocity vector. The first guess at the thrust attitude, however, is 20° away from this direction. Figure 1 illustrates the thrust attitude histories along successive trials. The discontinuity in slope at a velocity of 2000 fps is due to the fact that the guidance parameters are not recomputed beyond this point. It is not possible to continue the closed-loop calculations to the end of the trajectory, because excessively large control changes would be called for to correct small errors in the terminal quantity. In Table 1, the improvement in pay-off and the error in the terminal constraint are shown. It is seen that most of the improvement is accomplished on the first trial by just coming close to meeting the terminal constraint. After three trials the terminal mass is quite close to the optimum, although the thrust attitude is as much as 5° away from the optimum. Further improvement in thrust attitude produces very little change in terminal mass.

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Shearing Flow of a Viscoelastic Fluid between Porous Coaxial Cylinders

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THE problem of a Couette-type flow of a viscoelastic fluid between parallel walls, of which the fixed one is porous, recently has been investigated by the author.[1] In the present note, the related problem of shearing flow of a viscoelastic fluid between two coaxial cylinders, in which the outer one is moving parallel to its axis and the inner fixed cylinder is porous, is considered. This problem in the case of viscous fluid has been discussed by Dunwoody.[2]

The stress-strain relations for an incompressible viscoelastic fluid are given as

$$\sigma_j^i + \tau\tilde{\sigma}_j^i = 2\mu e_j^i \tag{1}$$

where σ_j^i is the extra-stress tensor, τ the elastic constant, μ the coefficient of viscosity, and e_j^i the rate of strain tensor. The term $\tilde{\sigma}_j^i$ appearing in Eq. (1) denotes its rate of change, which, following Truesdell,[3] one takes as

$$\tilde{\sigma}_j^i = (\partial\sigma_j^i/\partial t) + \sigma_{j,k}^i v^k + \sigma_j^i v_{,k}^k - \sigma^{ik} v_{j,k} - \sigma_j^k v_{,k}^i \tag{2}$$

Measuring z coordinate along the common axis of the cylinders and assuming axial symmetry, the equations of motion and continuity governing the problem are

$$\rho\left[u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z}\right] = -\frac{\partial p}{\partial r} + \frac{\partial\sigma_r^r}{\partial r} + \frac{\partial\sigma_z^r}{\partial z} + \frac{\sigma_r^r - \sigma_\theta^\theta}{r} \tag{3}$$

$$\rho\left[u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z}\right] = -\frac{\partial p}{\partial z} + \frac{\partial\sigma_z^r}{\partial r} + \frac{\partial\sigma_z^z}{\partial z} + \frac{\sigma_z^r}{r} \tag{4}$$

and

$$(\partial u/\partial r) + (u/r) + (\partial w/\partial z) = 0 \tag{5}$$

respectively.

If one further assumes the velocities to be functions of r only, Eqs. (3–5) reduce to

$$\rho u\frac{du}{dr} = -\frac{\partial p}{\partial r} + \frac{d\sigma_r^r}{dr} + \frac{\sigma_r^r - \sigma_\theta^\theta}{r} \tag{6}$$

$$\rho u\frac{dw}{dr} = -\frac{\partial p}{\partial z} + \frac{d\sigma_z^r}{dr} + \frac{\sigma_z^r}{r} \tag{7}$$

$$(1/r)(d/dr)(ru) = 0 \tag{8}$$

Also, Eq. (1), giving the stress-strain relation, becomes

$$\sigma_r^r + \tau\left[u\frac{d\sigma_r^r}{dr} - 2\sigma_r^r\frac{du}{dr}\right] = 2\mu\frac{du}{dr} \tag{9}$$

$$\sigma_z^r + \tau\left[u\frac{d\sigma_z^r}{dr} - \sigma_r^r\frac{dw}{dr} - \sigma_z^r\frac{du}{dr}\right] = \mu\frac{dw}{dr} \tag{10}$$

$$\sigma_z^z + \tau\left[u\frac{d\sigma_z^z}{dr} - 2\sigma_z^r\frac{dw}{dr}\right] = 0 \tag{11}$$

The boundary conditions of the problem are

$$u = U_0 \qquad w = 0 \text{ at } r = R_1 \tag{12}$$
$$w = U \text{ at } r = R_2$$

From Eqs. (8) and (12), one has $ru = \text{const} = R_1U_0$. This

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